The Magnetorotational Instability in the Kerr Metric

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ABSTRACT

The magnetorotational instability (MRI) is the leading candidate for driving turbulence, angular momentum transport, and accretion in astrophysical disks. I consider the linear theory of the MRI in a thin, equatorial disk in the Kerr metric. I begin by analyzing a mechanical model for the MRI that consists of two point masses on nearly circular orbits connected by a spring. I then develop a local Cartesian coordinate system for thin, equatorial Kerr disks. In this local model general relativistic effects manifest themselves solely through changes in the Coriolis parameter and in the tidal expansion of the effective potential. The MRI can be analyzed in the context of the local model using nonrelativistic magnetohydrodynamics, and the growth rates agree with those found in the mechanical model. The maximum growth rate measured by a circular orbit observer differs from a naive estimate using Newtonian gravity by a factor that varies between 1 and 4/3 for all radii and for all a/M.

Subject headings: accretion, accretion disks, black hole physics, Magnetohydrodynamics: MHD

1. Introduction

Balbus & Hawley (1991; hereafter BH) discovered a local instability of weakly magnetized accretion disks, studied earlier in the context of magnetized Couette flow by Velikhov (1959) and others. This instability is widely known as the magnetorotational instability (MRI). The instability conditions (see Balbus & Hawley 1998) are easily satisfied, so almost every ionized astrophysical disk is likely subject to the instability. Numerical experiments

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have demonstrated that the MRI can initiate and sustain magnetohydrodynamic (MHD) turbulence in disks (e.g. Hawley, Gammie, & Balbus 1995). They have also shown that MHD turbulence in disks transports angular momentum outward. MRI initiated turbulence is therefore a plausible (arguably the most plausible) candidate for driving accretion in astrophysical disks.

Disks around black holes are also likely subject to the MRI. The development of the MRI in these disks is of fundamental interest. But it is also of practical interest because numerical models of relativistic disks are now being developed (De Villiers & Hawley 2003; Gammie, McKinney, & Tóth 2003; De Villiers, Hawley, & Krolik 2003; Gammie, Shapiro, & McKinney 2003). In particular, one would like to know if the maximum growth rate is very different from the $(3/4)(GM/r^3)^{1/2}$ expected from analyses of Newtonian disks.

In this short paper I will study the MRI in thin, equatorial disks in the Kerr metric. I will use units such that $GM = c = 1$, where M is the black hole mass, and Boyer-Lindquist coordinates t, r, θ, ϕ, where

$$ds^2 = -\left(1 - \frac{2\,r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + \frac{A \sin^2\theta}{\Sigma}\, d\phi^2 - \frac{4\,a\,r \sin^2\theta}{\Sigma}\, d\phi\, dt, \tag{1}$$

and $\Sigma \equiv r^2 + a^2\cos^2\theta$, $\Delta \equiv r^2 - 2r + a^2$ and $A \equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$. It is useful to define the following relativistic correction factors that asymptote to 1 at large distance from the black hole:

$$\mathcal{A} \equiv 1 + a^2/r^2 + 2a^2/r^3, \tag{2}$$

$$\mathcal{B} \equiv 1 + ar^{-3/2}, \tag{3}$$

$$\mathcal{C} \equiv 1 - 3/r + 2ar^{-3/2}, \tag{4}$$

$$\mathcal{D} \equiv 1 - 2/r + a^2/r^2, \tag{5}$$

$$\mathcal{F} \equiv 1 - 2ar^{-3/2} + a^2/r^2, \tag{6}$$

and

$$\mathcal{G} \equiv 1 - 2/r + ar^{-3/2}. \tag{7}$$

These definitions are identical to those used in Novikov & Thorne (1973).

In what follows I will first develop a relativistic mechanical model for the MRI (§2), then a local Cartesian coordinate system that permits the treatment of the MRI near a rotating black hole using nonrelativistic MHD (§3). I compare the maximum growth of the instability to the shear rate in §4, and give a brief summary and prospects for future analysis in §5.

2. A Mechanical Analogy

BH considered the linear theory of a magnetized disk in the WKB approximation. The full theory is rather complicated, but Balbus & Hawley (1992) developed a useful mechanical model for the instability that consists of two point masses on nearly circular orbit connected by a spring. The masses are analogous to fluid elements connected by a magnetic field, which behaves like a spring. This mechanical model captures key features of the full MHD instability, including the maximum growth rate and the fact that the instability grows by exchange of angular momentum. Indeed, when both the magnetic field and the wavevector of the perturbation are parallel to the axis of rotation the mechanical modes have frequencies which are *identical* to the WKB modes of the full MHD model if one replaces the natural frequency γ of the spring-mass system by the Alfvén frequency $|\mathbf{k}\cdot\mathbf{v}_{\mathrm{A}}|$. It is therefore natural to begin a study of the MRI in the Kerr metric by considering a relativistic version of the Balbus & Hawley (1992) mechanical model.

The orbits of free particles are described by the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \tag{8}$$

which has the following solution for circular equatorial orbits:

$$u^\mu = \frac{dx^\nu}{d\tau} = \{\mathcal{B}\mathcal{C}^{-1/2}, 0, 0, r^{-3/2}\mathcal{C}^{-1/2}\} \tag{9}$$

$$= u^t\{1, 0, 0, \Omega\} \tag{10}$$

where $\Omega = 1/(r^{3/2}+a) = r^{-3/2}\mathcal{B}^{-1}$.

Consider perturbations about the circular orbit, $x^\mu \to x^\mu + \xi^\mu$, where ξ^μ is small. Then

$$\frac{d^2\xi^\mu}{d\tau^2} = -\partial_\alpha\Gamma^\mu_{\nu\lambda}u^\nu u^\lambda\xi^\alpha - 2\Gamma^\mu_{\nu\lambda}u^\nu\xi^\lambda + \mathcal{O}(|\xi|^2). \tag{11}$$

Substituting $\xi^\mu \sim \exp(-i\omega\tau)$ and evaluating the connection coefficients in Boyer-Lindquist coordinates, one finds

$$\omega^4(\omega^2-\nu^2)(\omega^2-\kappa^2) = 0. \tag{12}$$

Here

$$\nu^2 = \frac{1}{r^3}\frac{1-4a/r^{3/2}+3a^2/r^2}{\mathcal{C}} \tag{13}$$

is the "vertical" frequency and

$$\kappa^2 = \frac{1}{r^3}\frac{1-6/r+8ar^{-3/2}-3a^2/r^2}{\mathcal{C}} \tag{14}$$

is the epicyclic frequency. On the innermost stable circular orbit (ISCO) $\kappa^2 = 0$. The other four modes are zero-frequency relabelings of the t and ϕ coordinate.

The frequencies ν and κ are defined with respect to the proper time τ; they can be converted to frequencies with respect to the Boyer-Lindquist time coordinate t (and with respect to observers at infinity) by dividing by $u^t = dt/d\tau = \mathcal{B}\mathcal{C}^{-1/2}$ for a circular orbit. Using subscript t to denote a frequency with respect to t,

$$\kappa_t^2 = \Omega^2(1 - 6/r + 8ar^{-3/2} - 3a^2/r^2) \tag{15}$$

$$\nu_t^2 = \Omega^2(1 - 4ar^{-3/2} + 3a^2/r^2). \tag{16}$$

Now consider two particles tied together by a spring. The particles lie at $x^\mu \pm \xi^\mu$, and in the absence of rotation the natural frequency of the masses and spring is γ. The particles orbits now interact with the potential

$$\frac{1}{2}\gamma^2 h_{\mu\nu}\xi^\mu\xi^\nu \tag{17}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects the displacements into the space normal to the unperturbed (circular) orbit's four-velocity.

The equations of motion can be derived in a variety of ways, perhaps most simply by varying the action with respect to ξ^μ. The result is

$$\frac{d^2\xi^\mu}{d\tau^2} = -\partial_\alpha\Gamma^\mu_{\nu\lambda}u^\nu u^\lambda\xi^\alpha - 2\Gamma^\mu_{\nu\lambda}u^\nu\xi^\lambda - \gamma^2 h^\mu_\nu\xi^\nu. \tag{18}$$

Assuming that $\xi^\mu \sim \exp(-i\omega\tau)$, one finds that the nontrivial frequencies that correspond to motions in the equatorial plane obey

$$\omega^4 - \omega^2(2\gamma^2 + \kappa^2) + \gamma^2(\gamma^2 - s^2) = 0 \tag{19}$$

where

$$s^2 = \frac{3}{r^3}\frac{\mathcal{D}}{\mathcal{C}}. \tag{20}$$

Equation (19) is quadratic in ω^2 and so easily solved. By varying γ one finds that the maximum growth rate is

$$-\omega^2_{max} = \frac{s^4}{4(s^2+\kappa^2)} = \frac{9}{16}\frac{1}{r^3}\left(\frac{\mathcal{D}}{\mathcal{C}}\right)^2 \tag{21}$$

which occurs for

$$\gamma^2 = \frac{s^2}{4}\frac{s^2 + 2\kappa^2}{s^2+\kappa^2}. \tag{22}$$

The maximum growth rate is defined with respect to proper time τ; it is the growth rate measured by an observer on a circular orbit at radius r. Converting to frequencies with respect to Boyer-Lindquist time t (observers at infinity):

$$-\omega_{max,t}^2 = \frac{9}{16}\Omega^2\left(\frac{\mathcal{D}^2}{\mathcal{C}}\right) \tag{23}$$

This may be compared to the Newtonian maximum growth rate $-\omega_{max,N}^2 = (9/16)r^{-3}$. I will refer to the ratio of the relativistic maximum growth rate to Newtonian maximum growth rate $f \equiv |\omega_{max}/\omega_{max,N}|$ as the *normalized growth rate.*

Figure 1 shows the run of the normalized growth rate for $a = 0.9, 0$, and -0.9 (i.e. a counterrotating disk). The most dramatic effect is the drop in normalized growth rate with respect to time t in the prograde disk. This is due to gravitational redshift (the conversion from time τ to time t) so it is a global feature of the spacetime rather than a characteristic of the MRI: any disk instability would experience the same redshift. The effect is strongest on the ISCO for prograde disks. The normalized growth rate with respect to t drops to zero on the ISCO as $a \to 1$, since $u^t \to \infty$ there (see eq. 9).

The very limited variation of normalized growth rate with respect to proper time τ as r and a vary *is* peculiar to the MRI. This can be understood as follows. Since $-\omega_{max}^2 = s^4/(4(s^2+\kappa^2))$ and $\kappa^2 = 0$ at the ISCO, $-\omega_{max}^2 = s^2/4$ at the ISCO. One may also show that $s^2 + \kappa^2 = 4r^{-3}$, hence $-\omega_{max}^2 = r^{-3}$ at the ISCO. Then $\omega_{max}^2/\omega_{max,N}^2 = 16/9$ on the ISCO. Since the relativistic correction to the MRI growth rate is largest at the ISCO, the normalized growth rate varies between 4/3 at $r = r_{ISCO}$ and 1 as $r \to \infty$.

3. Local Model

In studies of thin disks in a Newtonian potential it has proven useful to introduce a local Cartesian coordinate frame, or *local model.* This frame is centered on a circular orbit, the x axis points along the radius vector, the y axis points forward in azimuth, and the z axis points up, normal to the disk. Beginning with a global spherical coordinate system r, θ, ϕ, the transformation to nonrelativistic local coordinates is

$$x = r - r_0 \tag{24}$$

$$y = r_0(\phi - \phi_0 - \Omega t) \tag{25}$$

where ϕ_0 is a constant and here only $\Omega = r_0^{-3/2}$, and

$$z = r_0(\frac{\pi}{2} - \theta). \tag{26}$$

Expanding to lowest order in $|\mathbf{x}|/r_0 \sim |\dot{\mathbf{x}}|/r_0\Omega$, the free particle equations of motion are

$$\ddot{x} = 2\Omega\dot{y} + 3\Omega^2 x, \tag{27}$$

$$\ddot{y} = -2\Omega\dot{x}, \tag{28}$$

$$\ddot{z} = -\Omega^2 z. \tag{29}$$

The linear theory of magnetized disks in the WKB approximation can be carried over in its entirety to this local model because higher order terms in ϵ are also higher order terms in the WKB approximation.

In a relativistic disk in the equatorial plane of the Kerr metric we may proceed in exactly the same way. We want to erect local Cartesian coordinates near a circular orbit. In Boyer-Lindquist coordinates the circular orbit is located at $r = r_0$ and has $d\phi/dt = \Omega_0 = 1/(r_0^{3/2} + a)$. The transformation to local coordinates τ, x, y, z is

$$\tau = t\mathcal{G}\mathcal{C}^{-1/2} - (\phi - \phi_0) r_0^{1/2} \mathcal{F}\mathcal{C}^{-1/2} \tag{30}$$

$$x = (r - r_0)\mathcal{D}^{-1/2}, \tag{31}$$

$$y = r_0(\phi - \phi_0 - \Omega_0 t)\mathcal{B}\mathcal{D}^{1/2}\mathcal{C}^{-1/2} \tag{32}$$

$$z = r_0(\frac{\pi}{2} - \theta). \tag{33}$$

This transformation is determined by the conditions that the local coordinates should be locally Minkowski, and that the point $x, y, z = 0$ should follow a circular orbit. This transformation is (not by accident) identical to the transformation to a corotating orthonormal basis of one-forms given by Novikov & Thorne (1973).

Now consider the limit $\epsilon^2 \equiv (x^2 + y^2 + z^2)/r_0^2 \ll 1$. Expanding g_{tt} to $\mathcal{O}(\epsilon^2)$, g_{ti} to $\mathcal{O}(\epsilon)$ and g_{ij} to $\mathcal{O}(1)$ (higher order terms are not required for a consistent expansion of ds^2; one can think of $d\tau$ as being $\mathcal{O}(1)$ and dx^i as being $\mathcal{O}(\epsilon)$), the metric becomes

$$ds^2 = (-1 + s^2 x^2 - \nu^2 z^2) d\tau^2 + 4\tilde{\Omega} x d\tau dy + dx^2 + dy^2 + dz^2 \tag{34}$$

where $\tilde{\Omega} = r_0^{-3/2}$,

$$s^2 = \frac{3}{r_0^3}\left(\frac{\mathcal{D}}{\mathcal{C}}\right), \tag{35}$$

and

$$\nu^2 = \frac{1}{r_0^3}\left(\frac{1 - 4ar_0^{-3/2} + 3a^2/r_0^2}{\mathcal{C}}\right). \tag{36}$$

These are the same as the s and ν defined in the last section.

The metric component $g_{\tau\tau}$ can be interpreted as $\approx -1 + 2\psi$ where ψ is the effective gravitational potential, and the $g_{\tau y}$ term indicates that the local model is a frame rotating with frequency $\tilde{\Omega}$. Using this interpretation, or working directly with the geodesic equation, one arrives at the equations of motion for a free particle

$$\ddot{x} = 2\tilde{\Omega}\dot{y} + s^2 x, \tag{37}$$

$$\ddot{y} = -2\tilde{\Omega}\dot{x}, \tag{38}$$

$$\ddot{z} = -\nu^2 z, \tag{39}$$

where we have assumed that $(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/(\tilde{\Omega} r_0^2) \sim \epsilon^2$. One can look for motions with time dependence $\exp(-i\omega\tau)$; the free particle frequencies are identical to those found from the perturbed geodesic equation in §2.

If the disk is thin then the sound speed is small compared to the speed of light. Since one expects the fluid speeds within the disk to be subsonic, the fluid motions are nonrelativistic. It follows that in the local Cartesian frame the fluid obeys the equations of nonrelativistic fluid dynamics. The MHD equations of motion in the local model are then

$$\frac{D\mathbf{v}}{Dt} = -2\tilde{\mathbf{\Omega}} \times \mathbf{v} + s^2 x \mathbf{e}_x - \nu^2 z \mathbf{e}_z - \frac{\nabla p}{\rho} - \frac{\nabla B^2}{8\pi\rho} + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi\rho}, \tag{40}$$

where $\tilde{\mathbf{\Omega}} \equiv \tilde{\Omega}\mathbf{e}_z$. These equations, coupled to the continuity equation, the (nonrelativistic) induction equation, an energy equation, and the condition $\nabla \cdot \mathbf{B} = 0$, give a complete description of the local evolution of the plasma in the MHD approximation.

The nonrelativistic linear analysis of the Balbus-Hawley instability can now be carried over into the local frame by replacing the Newtonian rotation frequency Ω with $\tilde{\Omega}$, the coefficient of the tidal expansion of the effective potential $(3/2)\Omega^2$ by s^2, and the Newtonian vertical frequency Ω by ν. General relativistic effects appear solely through the Coriolis parameter $\tilde{\Omega}$ and the tidal expansion parameter s.

To carry forward a linear analysis we need to specify an equilibrium. The simplest choice, which captures most of the features of the nonrelativistic instability, is a uniform vertical magnetic field near the midplane of the disk, so the Alfvén velocity can be written $\mathbf{v}_A = (B/(4\pi\rho)^{1/2})\mathbf{e}_z$. I will restrict consideration to perturbations of the form $\exp(-i\omega\tau + ik_z z)$ (more general perturbations do not introduce qualitatively new features in ideal MHD). Then the dispersion relation is

$$\omega^4 - \omega^2(\kappa^2 + 2(\mathbf{k}\cdot\mathbf{v}_A)^2) + (\mathbf{k}\cdot\mathbf{v}_A)^2((\mathbf{k}\cdot\mathbf{v}_A)^2 - s^2) = 0 \tag{41}$$

Since this is identical to the relation obeyed by the mechanical model, with $\gamma^2 \to (\mathbf{k} \cdot \mathbf{v}_A)^2$, the maximum growth rate of the MHD instability is the same as that of the mechanical model.

4. Relationship to Shear Rate

Balbus & Hawley (1992) conjectured that the maximum growth rate should always be half the shear rate. To check this, we need a relativistic generalization of the shear rate. The natural generalization involves the scalar

$$\sigma^2 = \sigma^{\mu\nu}\sigma_{\mu\nu} \tag{42}$$

where

$$\sigma_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha;\mu}h^{\mu}_{\beta} + u_{\beta;\mu}h^{\mu}_{\alpha}\right) - \frac{1}{3}\Theta h_{\alpha\beta} \tag{43}$$

is the rate-of-strain tensor and

$$\Theta \equiv u^{\alpha}{}_{;\alpha} \tag{44}$$

is the divergence of the velocity field. For a Cartesian shear flow $v_y = Ax$, where A is constant, one finds $\sigma^2 = A^2/2$, so $2\sigma^2$ is the square of the shear rate.

For circular equatorial geodesics in the Kerr metric σ^2 can be evaluated in any frame, including the Boyer-Lindquist coordinate frame (this is a lengthy calculation!). Novikov & Thorne (1973) evaluated $\sigma_{\mu\nu}$ in an orthonormal tetrad moving with the flow, which is equivalent to our local coordinate frame. They found

$$\sigma_{(r)(\phi)} = \sigma_{(\phi)(r)} = \frac{3}{4}\frac{1}{r^{3/2}}\left(\frac{\mathcal{D}}{\mathcal{C}}\right), \tag{45}$$

and all other components are zero. Then

$$\sigma^2 = \frac{9}{8}\frac{1}{r^3}\left(\frac{\mathcal{D}}{\mathcal{C}}\right). \tag{46}$$

Evidently

$$-\omega^2_{max} = \frac{1}{4}(2\sigma^2) \tag{47}$$

This confirms the relationship between the maximum growth rate and the shear rate suggested by Balbus & Hawley (1992). In the local coordinate system, of course, the relationship is trivially verified.

5. Conclusion

I have shown that the growth rate of the MRI in thin, equatorial disks around Kerr black holes does not differ sharply from its Newtonian counterpart, at least as measured by an observer on a circular orbit. As measured by an observer at infinity the growth rate is redshifted strongly for instabilities growing near the marginally stable orbit in corotating disks.

A similar analysis could be carried out for inclined disks. The work of Bardeen & Petterson (1975) and successors (see Demianski & Ivanov 1997 and references therein) suggests, however, that close to the hole disks settle into the equatorial plane of the black hole in the presence of angular momentum diffusion (such as that which might result from MHD turbulence initiated by the MRI). At larger distance the deviations from Newtonian gravity will be small and will likely not affect the development of the most unstable modes, which grow on the dynamical timescale.

Earlier work by Gammie & Popham (1998) (see footnote 7) stated the results presented here without proof. Araya-Góchez (2002) gives an impressive fully relativistic linear analysis that is equivalent to results presented here (compare Figure 1 with his Figure 2). The fully relativistic analysis is perhaps the ultimate proof that the MRI carries over nearly unchanged to relativistic disks; the present paper states these results in a greatly simplified form.

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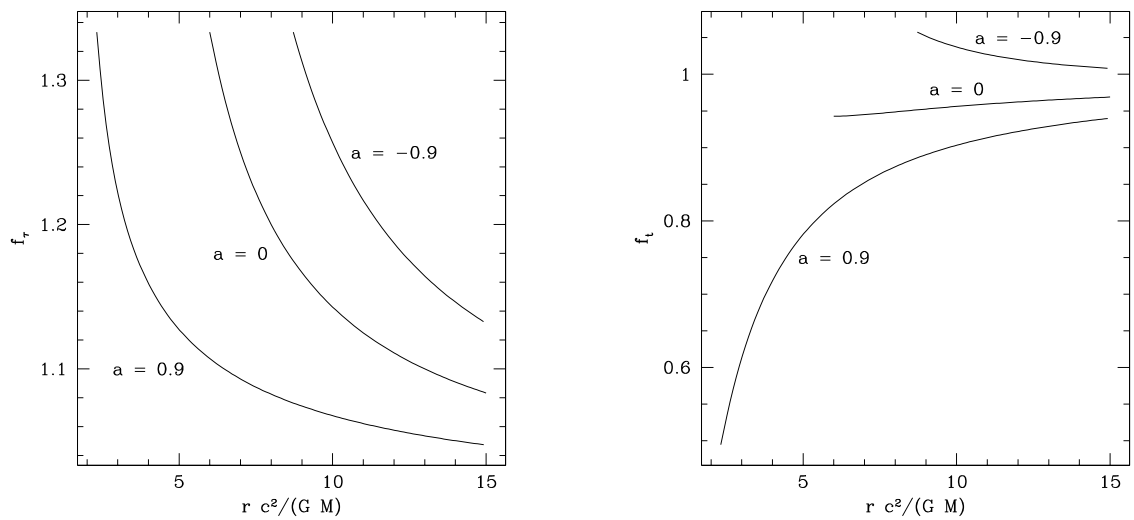

Fig. 1.— The figure shows the maximum growth rate of the Balbus-Hawley instability for a thin disk in the equatorial plane of the Kerr metric normalized to the naively calculated nonrelativistic growth rate. The left panel shows f_τ, the normalized growth rate with respect to the proper time τ of an observer on a circular orbit. The right panel shows f_t, the normalized growth rate with respect to the Boyer-Lindquist time t, which is the growth rate measured by an observer at large radius.